

## AC Circuits - Notes - Part 2

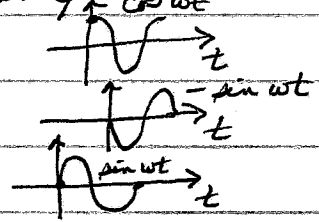
Phase angle of a cosine or sine wave:

Consider  $\cos(\omega t + \phi) = \cos \omega t \cos \phi - \sin \omega t \sin \phi$

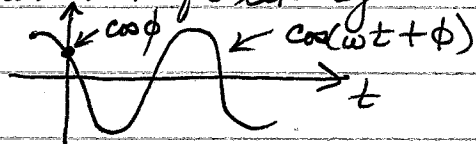
$$\phi = 0, \cos(\omega t + 0) = \cos \omega t$$

$$\phi = \frac{\pi}{2}, \cos(\omega t + \frac{\pi}{2}) = -\sin \omega t$$

$$\phi = -\frac{\pi}{2}, \cos(\omega t - \frac{\pi}{2}) = \sin \omega t$$



So the wave shifts laterally when the phase angle  $\phi \neq 0$ .

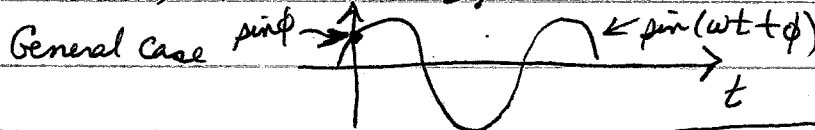
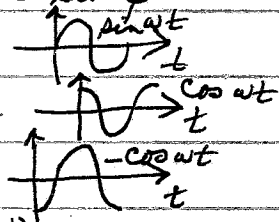


Similarly,  $\sin(\omega t + \phi) = \sin \omega t \cos \phi + \cos \omega t \sin \phi$

$$\phi = 0, \sin(\omega t + 0) = \sin \omega t$$

$$\phi = \frac{\pi}{2}, \sin(\omega t + \frac{\pi}{2}) = \cos \omega t$$

$$\phi = -\frac{\pi}{2}, \sin(\omega t - \frac{\pi}{2}) = -\cos \omega t$$



We want to generalize Ohm's Law to  $V = IZ$ , where  $Z = \text{complex impedance}$

Capacitor  $Q = CV$ ,  $I = C \frac{dV}{dt}$ , Inductor  $V = L \frac{dI}{dt}$

$$V_C = \frac{1}{C} \int I dt$$

Suppose  $I = I_0 \cos \omega t = I_0 \operatorname{Re} [e^{i\omega t}] = I_0 \operatorname{Re} [\cos \omega t + i \sin \omega t]$

$$\textcircled{1} \quad V_C = \frac{1}{C} \int I dt = \frac{I_0}{C} \int \cos \omega t dt = \frac{I_0}{\omega C} \sin \omega t$$

$$\textcircled{2} \quad V_L = L \frac{dI}{dt} = L I_0 \omega \sin \omega t$$

Now use complex algebra instead of calculus.  $Z_C = \frac{1}{i\omega C}$ ,  $Z_L = i\omega L$

Represent  $I$  as  $\operatorname{Re} [\text{function}]$ , do Ohm's Law with complex numbers,

and take Re part at end to get what we actually measure on oscilloscope.

$$V_L = I(i\omega L) = I_0 i\omega L e^{i\omega t} = I_0 L \omega i(\cos \omega t + i \sin \omega t)$$

$$= I_0 L \omega (i \cos \omega t - \sin \omega t)$$

Take Re part to get  $V_L = -I_0 L \omega \sin \omega t$ , same as  $\textcircled{2}$

$$V_C = I Z_C = I_0 \frac{1}{i\omega C} (e^{i\omega t}) = \frac{I_0}{i\omega C} (\cos \omega t + i \sin \omega t)$$

$$= \frac{I_0}{\omega C} (-i \cos \omega t + \sin \omega t)$$

Take Re part to get  $V_C = \frac{I_0}{\omega C} \sin \omega t$ , same as  $\textcircled{1}$

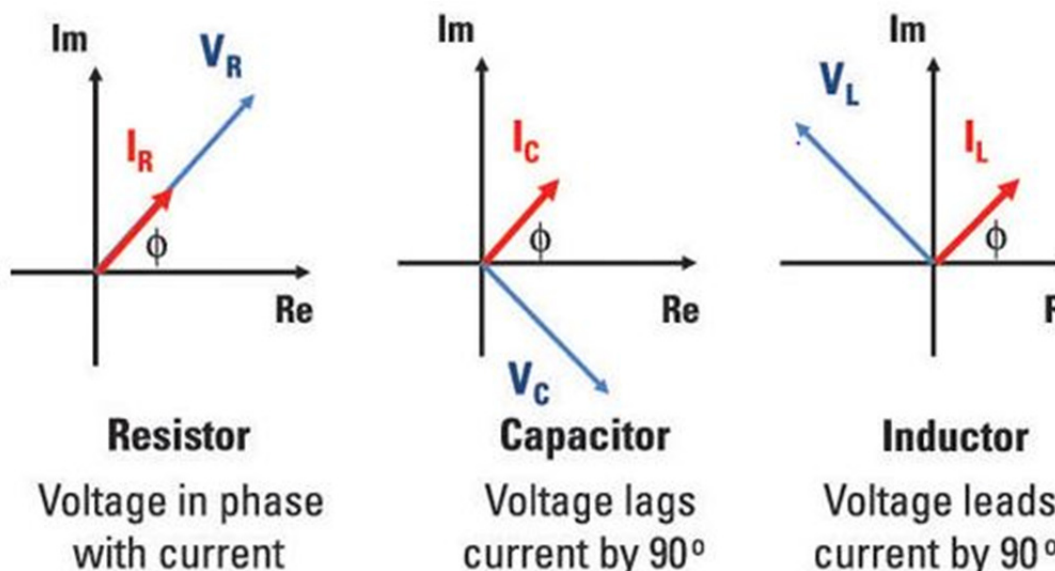
Thus, if we define the complex impedance,  $Z$ , properly ( $Z_R = R$  for resistor,  $Z_C = \frac{1}{i\omega C}$  for capacitor, and  $Z_L = i\omega L$  for an inductor) in our generalization of Ohm's Law,  $V=IZ$ , we can use complex algebra instead of solving differential equations!

Because we have generalized Ohm's Law, and Kirchhoff's Laws still hold for AC circuits (sum of currents coming into node = sum of currents leaving, and voltages sum to zero going around a loop), the equations for adding resistors in series and in parallel all hold for complex impedances.

i.e.,  $Z_{\text{Series}} = Z_1 + Z_2 + Z_3 + \dots + Z_N$ , for  $N$  complex impedances in series

and  $\frac{1}{Z_{\text{Parallel}}} = \frac{1}{Z_1} + \frac{1}{Z_2} + \frac{1}{Z_3} + \dots + \frac{1}{Z_N}$ , for  $N$  complex impedances in parallel.

An equivalent way to think about AC circuit problems is to note that the differential equations given on the previous page ( $I = C \frac{dV}{dt}$  and  $V = L \frac{dI}{dt}$ ) indicate that the voltage and current in capacitors and inductors are each out of phase by  $90^\circ$ . For an inductor, the voltage leads the current by  $90^\circ$ , whereas for a capacitor, the voltage lags behind the current by  $90^\circ$ . (See the equations on the previous page to see that these statements are true.) These relationships can be summarized by the following diagram from *Circuit Analysis for Dummies* (see <https://www.dummies.com/education/science/science-electronics/generalize-impedance-to-expand-ohms-law-to-capacitors-and-inductors/>)



This explains why many undergraduate lower division textbooks do circuit analysis by adding phasors (similar to adding vectors). If you use complex numbers instead, however, keeping track of the vector components is done for you by doing complex algebra (since a complex number  $z$  can be written as  $x+iy$ , and plotted in the complex plane). I personally find complex algebra to be easier than adding vectors!