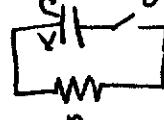


10/30/06

Discharging capacitorC initially charged to voltage  $V_0$ . Switch closed at  $t=0$ .

$$Q = CV, \quad I = \frac{V}{R}, \quad I = -\frac{dQ}{dt} = \frac{CV_0 e^{-t/RC}}{RC}$$

$$\Rightarrow \frac{dQ}{dt} = -\frac{Q}{RC}$$

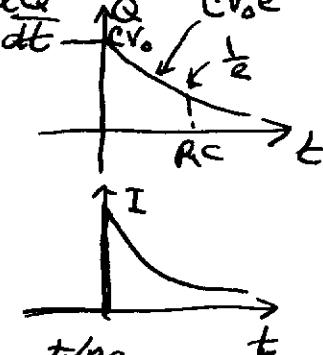
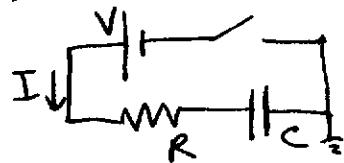
$$\frac{dQ}{Q} = -\frac{dt}{RC}$$

$$\ln Q = -\frac{t}{RC} + \text{Const}$$

$$Q = A_0 e^{-t/RC}$$

$$\text{At } t=0, V=V_0, Q=CV_0 \Rightarrow Q = CV_0 e^{-t/RC}$$

$$I = -\frac{dQ}{dt} = \frac{V_0}{RC} e^{-t/RC}$$

RC Circuit with batteryClose switch at  $t=0$ . C is initially uncharged.

$$V = IR + \frac{Q}{C} = R\frac{dQ}{dt} + \frac{Q}{C} \quad \begin{matrix} \text{inhomogeneous} \\ \text{inhomogeneous term} \end{matrix}$$

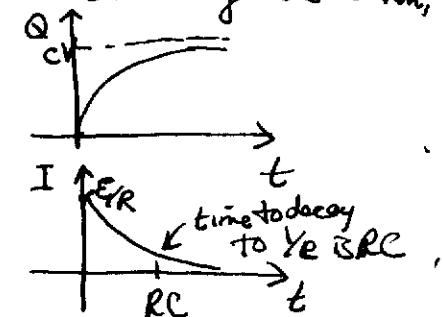
Solution is homogeneous solution above + inhomogeneous solution:

$$Q = A_0 e^{-t/RC} + CV \quad \begin{matrix} \text{homogeneous solution} \\ \text{inhomogeneous solution} \end{matrix}$$

$$\text{At } t=0, Q=0 = A_0 + CV \Rightarrow A_0 = -CV$$

$$\therefore Q = CV(1 - e^{-t/RC})$$

$$I = \frac{dQ}{dt} = \frac{V}{RC} e^{-t/RC}$$

Note at  $t \rightarrow \infty$ ,  $Q=CV$ ,  $I=0$ , as expected.

## Review of complex numbers

$$i = \sqrt{-1}, i^2 = -1, \frac{1}{i} = -i$$

$$z = x + iy, z^* = x - iy, |z| = \sqrt{zz^*} = \sqrt{x^2 + y^2}$$

$$z = |z| e^{i\theta}, \tan \theta = \frac{y}{x}, e^{i\theta} = \cos \theta + i \sin \theta$$

$$z = \frac{A+Bi}{C+iD}, z^* = \frac{A-Bi}{C-iD}$$

$$C+iD, C-iD$$

## Sec 8.3 AC Circuits (quoting from E. M. Purcell, Electricity and Magnetism, 2nd. Ed, Sect. 8.3)

1. An alternating current or voltage can be represented by a complex number.
2. Any one branch or element of the circuit can be characterized, at a given frequency, by the relation between the voltage and the current in that branch.

RECALL:  $e^{i\theta} = \cos \theta + i \sin \theta$ , where  $i = \sqrt{-1}$ .

Adopt the following rule for the representation:

1. An alternating current  $I_o \cos(\omega t + \phi)$  is to be represented by the complex number  $I_o e^{i\phi}$ , that is, the number whose real part is  $I_o \cos \phi$  and whose imaginary part is  $I_o \sin \phi$ .
2. Going the other way, if the complex number  $x + iy$  represents a current  $I$ , then the current as a function of time is given by the real part of the product  $(x + iy)e^{i\omega t}$ .

$$I = YV \quad V = IZ$$

Circuit Element	Admittance $Y$	Impedance $Z$
Resistor	$\frac{1}{R}$	R
Inductor	$\frac{1}{i\omega L}$	$i\omega L$
Capacitor	$i\omega C$	$\frac{1}{i\omega C} = \frac{-i}{\omega C}$

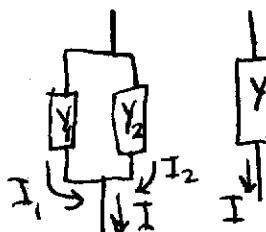
Representation of the sum of 2 currents is the sum of the representations.

$$\textcircled{1} \quad I_1 + I_2 = I_{o1} \cos(\omega t + \phi_1) + I_{o2} \cos(\omega t + \phi_2)$$

$$= (I_{o1} \cos \phi_1 + I_{o2} \cos \phi_2) \cos \omega t - (I_{o1} \sin \phi_1 + I_{o2} \sin \phi_2) \sin \omega t$$

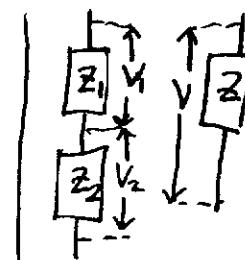
$$\textcircled{2} \quad I_{o1} e^{i\phi_1} + I_{o2} e^{i\phi_2} = (I_{o1} \cos \phi_1 + I_{o2} \cos \phi_2) + i(I_{o1} \sin \phi_1 + I_{o2} \sin \phi_2)$$

If you multiply  $\textcircled{2}$  by  $e^{i\omega t} = \cos \omega t + i \sin \omega t$  and take the real part of the result, you will obtain  $\textcircled{1}$ .



Admittances add in parallel

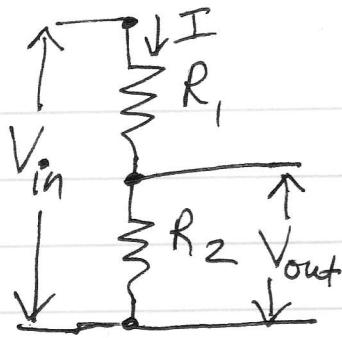
$$I = I_1 + I_2 = Y_1 V + Y_2 V = (Y_1 + Y_2) V = YV$$



Impedances add in series.

$$V = V_1 + V_2 = IZ_1 + IZ_2 = I(Z_1 + Z_2) = IZ$$

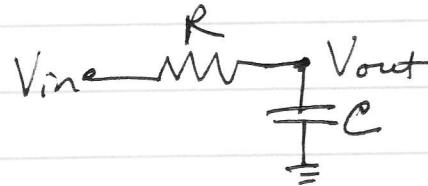
Voltage divider



$$I = \frac{V_{in}}{R_1 + R_2} = \frac{V_{out}}{R_2}$$

$$V_{out} = V_{in} \frac{R_2}{R_1 + R_2}$$

Low-pass filter



$$V_{out} = V_{in} \frac{\frac{1}{i\omega C}}{R + \frac{1}{i\omega C}} = \frac{V_{in}}{i\omega RC + 1}$$

$$\omega = 0, V_{out} = V_{in}$$

$$\omega = \infty, V_{out} = 0$$

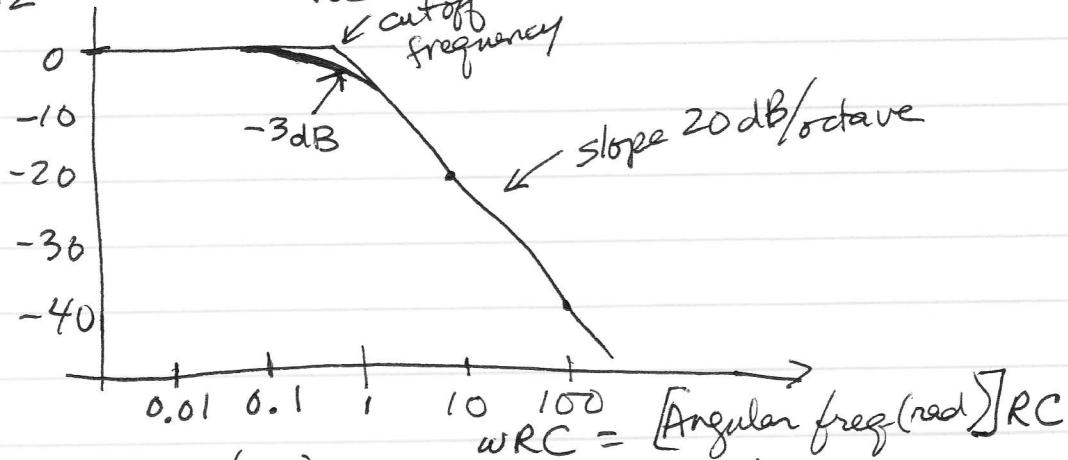
Bandwidth = range of frequencies passed  
Cutoff freq when signal attenuated  
to  $\frac{1}{2}$  power  $\Rightarrow G_C = \frac{1}{\sqrt{2}}$

$$G_C = \frac{|V_{out}|}{|V_{in}|} = \frac{1}{\sqrt{1 + \omega^2 R^2 C^2}}$$

$$G_C = \frac{1}{\sqrt{2}} \text{ when } \omega = \frac{1}{RC}$$

Bode plot

Gain (dB)

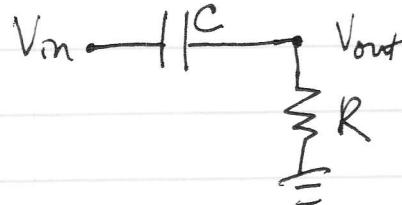


$$\omega RC = [\text{Angular freq (rad)}] RC$$

$$3 \text{ dB} = 10 \log_{10} 2$$

$$\Rightarrow 20 \text{ dB} = 20 \log_{10} 10$$

High-pass filter

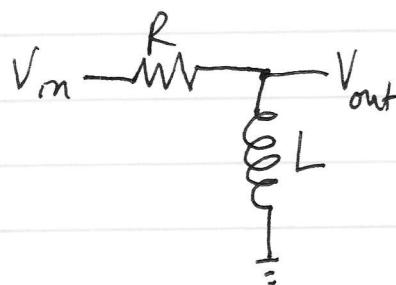


$$V_{out} = V_{in} \frac{R}{R + \frac{1}{i\omega C}} = \frac{V_{in} R}{i\omega RC + 1}$$

$$\omega = 0, V_{out} = 0$$

$$\omega = \infty, V_{out} = V_{in}$$

RL High-pass filter



RL Low-pass filter



12/1/06. Parallel RLC Circuit

$$E_{\text{court}} \odot \begin{array}{c} \text{---} \\ | \\ \frac{1}{C} \end{array} \parallel \begin{array}{c} \text{---} \\ | \\ R \end{array} \parallel \begin{array}{c} \text{---} \\ | \\ L \end{array} \quad Y = \frac{1}{R} + i\omega C + \frac{1}{i\omega L} \quad I = YV = E_0 \left[ \frac{1}{R} + i\left(\omega C - \frac{1}{\omega L}\right) \right]$$

$$\text{Amplitude of current oscillation} = |I| = \sqrt{II^*} = E_0 \left[ \frac{1}{R^2} + \left(\omega C - \frac{1}{\omega L}\right)^2 \right]^{1/2}$$

$$\tan \varphi = \frac{\omega C - \frac{1}{\omega L}}{\frac{1}{R}} = R\omega C - \frac{R}{\omega L}, \quad I = |I| e^{i\varphi}$$

ex. Series RLC Circuit

$$I_{\text{court}} \odot \begin{array}{c} \text{---} \\ | \\ R \end{array} \parallel \begin{array}{c} \text{---} \\ | \\ C \end{array} \parallel \begin{array}{c} \text{---} \\ | \\ L \end{array} \quad Z = R + i\omega L + \frac{1}{i\omega C} \quad V = I_0 Z = I_0 \left[ R + i\left(\omega L - \frac{1}{\omega C}\right) \right] \quad |V| = \sqrt{VV^*} = I_0 \left[ R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2 \right]^{1/2} \quad \tan \varphi = \frac{\omega L - \frac{1}{\omega C}}{R} = \frac{\omega L}{R} - \frac{1}{\omega CR}, \quad V = |V| e^{i\varphi}$$

### Ex. 8.5 Power and Energy in ac Circuits

If voltage across resistor  $R$  is  $V_0 \cos \omega t$ ,  $P = \frac{V^2}{R} = \frac{V_0^2 \cos^2 \omega t}{R}$   
 $\langle \cos^2 \omega t \rangle = \frac{1}{2}$ ,  $\bar{P} = \frac{1}{2} \frac{V_0^2}{R} = \frac{V_{\text{rms}}^2}{R}$ , where  $V_{\text{rms}} = \frac{V_0}{\sqrt{2}}$  root mean square

U.S. line voltage is 120 V rms, 60 Hz,

$$\text{i.e., } V(t) = 170 \cos \underbrace{377t}_{120\sqrt{2}} \quad \underbrace{2\pi(60)}$$

Instantaneous power delivered to circuit element is  $P = IV$ .

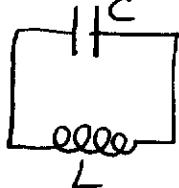
$$\text{Let } V = E_0 \cos \omega t, \quad I = I_0 \cos(\omega t + \varphi)$$

$$\text{Then } P = IV = E_0 I_0 \cos \omega t \cos(\omega t + \varphi)$$

$$= E_0 I_0 \left( \underbrace{\cos^2 \omega t \cos \varphi}_{\text{time average} = \frac{1}{2}} - \underbrace{\cos \omega t \sin \omega t \sin \varphi}_{\frac{1}{2} \sin 2\omega t \text{ has time average} = 0} \right)$$

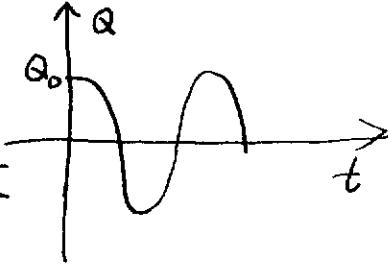
$$\bar{P} = \frac{1}{2} E_0 I_0 \cos \varphi = V_{\text{rms}} I_{\text{rms}} \cos \varphi \quad \begin{matrix} \uparrow \\ \text{power factor} \end{matrix}$$

11/27/06 L-C Circuit - Resonant Circuit



$$L \frac{dI}{dt} + \frac{Q}{C} = 0, \quad I = \frac{dQ}{dt}, \quad \frac{dI}{dt} = \frac{d^2Q}{dt^2}$$

$$\frac{d^2Q}{dt^2} + \frac{1}{LC} Q = 0$$



$$\text{Let } Q = Q_0 \cos(\omega t + \varphi)$$

$$\text{Then } \frac{dQ}{dt} = -Q_0 \omega \sin(\omega t + \varphi) = I$$

$$\frac{d^2Q}{dt^2} = -Q_0 \omega^2 \cos(\omega t + \varphi)$$

Substitute back into the differential equation:  $(-\omega^2 + \frac{1}{LC}) Q_0 \cos(\omega t + \varphi) = 0$

$$\Rightarrow \omega = \frac{1}{\sqrt{LC}}$$

$$\text{Total energy} = \frac{1}{2} LI^2 + \frac{1}{2} \frac{Q^2}{C} = \frac{1}{2} L Q_0^2 \omega^2 \sin^2(\omega t + \varphi) + \frac{1}{2} \frac{Q_0^2}{C} \cos^2(\omega t + \varphi) \\ = \frac{1}{2} \frac{Q_0^2}{C}$$

Energy oscillates between completely electric (capacitor fully charged) to completely magnetic (all energy in inductor).

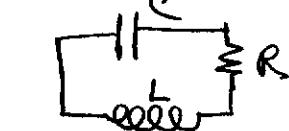
RLC Series Circuit - Resonant Circuit with damping

$$L \frac{dI}{dt} + IR + \frac{Q}{C} = 0, \quad I = \frac{dQ}{dt}, \quad \frac{dI}{dt} = \frac{d^2Q}{dt^2}$$

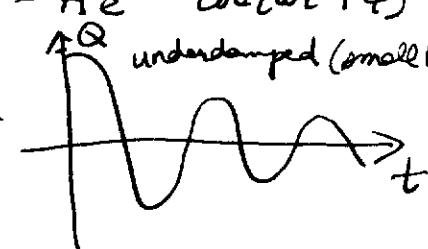
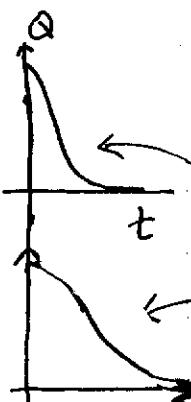
$$\frac{d^2Q}{dt^2} + \frac{R}{L} \frac{dQ}{dt} + \frac{1}{LC} Q = 0 \quad (\text{Same as Parallel R, with } Q = CV).$$

$$Q = A e^{-\frac{R}{2L}t} \cos\left(\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} t + \varphi\right) = A e^{-\frac{Rt}{2L}} \cos(\omega t + \varphi)$$

i.e. where  $\omega = \frac{1}{\sqrt{LC}} - \frac{R^2}{4L^2}$



underdamped (small R)



Critical damping,  $\omega = 0, R = 2\sqrt{\frac{L}{C}}$ .

$$Q(t) = (A + Bt) e^{-\beta t}$$

Overdamping,  $R > 2\sqrt{\frac{L}{C}}$ ,  $Q(t) = A e^{-\beta_1 t} + B e^{-\beta_2 t}$

AC Circuits  $I = I_0 \cos \omega t$

$$t \text{ Inductor: } V = L \frac{dI}{dt} = -L \omega I_0 \sin \omega t = -L \omega I_0 \cos(\omega t + \frac{\pi}{2})$$

$$\text{Capacitor: } Q = CV, I = \frac{dQ}{dt} = C \frac{dV}{dt}, V = \frac{1}{C} \int I dt = \frac{I_0}{\omega C} \sin \omega t = \frac{I_0}{\omega C} \cos(\omega t - \frac{\pi}{2})$$

$$\text{Resistor: } V = IR$$

